2.

The Language of First-order Logic

Declarative language

Before building system

before there can be learning, reasoning, planning, explanation ...

need to be able to express knowledge

Want a precise declarative language

- declarative: believe *P* = hold *P* to be <u>true</u>

  cannot believe *P* without some sense of
  what it would mean for the world to satisfy *P*

- precise: need to know exactly

  what strings of symbols count as sentences

  what it means for a sentence to be true

  (but without having to specify which ones are true)

Here: language of first-order logic

again: not the only choice

Alphabet

Logical symbols:

- Punctuation: (,), .
- Connectives: $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$, $=$
- Variables: $x, x_1, x_2, \ldots, x', x'', \ldots, y, \ldots, z, \ldots$

  Fixed meaning and use
  like keywords in a programming language

Non-logical symbols

- Predicate symbols (like Dog)

  Note: not treating = as a predicate
- Function symbols (like bestFriendOf)

  Domain-dependent meaning and use
  like identifiers in a programming language

Have <u>arity</u>: number of arguments

arity 0 predicates: propositional symbols

arity 0 functions: constant symbols

Assume infinite supply of every arity

Grammar

Terms

1. Every variable is a term.
2. If $t_1, t_2, \ldots, t_n$ are terms and f is a function of arity n, then $f(t_1, t_2, \ldots, t_n)$ is a term.

Atomic wffs (well-formed formula)

1. If $t_1, t_2, \ldots, t_n$ are terms and P is a predicate of arity n, then $P(t_1, t_2, \ldots, t_n)$ is an atomic wff.
2. If t_1 and t_2 are terms, then $(t_1{=}t_2)$ is an atomic wff.

Wffs

1. Every atomic wff is a wff.
2. If α and β are wffs, and v is a variable, then $\neg\alpha$, $(\alpha\wedge\beta)$, $(\alpha\vee\beta)$, $\exists v.\alpha$, $\forall v.\alpha$ are wffs.

The propositional subset: no terms, no quantifiers

Atomic wffs: only predicates of 0-arity: $(p \wedge \neg(q \vee r))$

Notation

Occasionally add or omit (,), .

Use [,] and {,} also.

Abbreviations:

$(\alpha \supset \beta)$ for $(\neg\alpha \vee \beta)$

safer to read as disjunction than as "if ... then ..."

$(\alpha \equiv \beta)$ for $((\alpha\supset\beta) \wedge (\beta\supset\alpha))$

Non-logical symbols:

- Predicates: mixed case capitalized

  Person, Happy, OlderThan

- Functions (and constants): mixed case uncapitalized

  fatherOf, successor,
  johnSmith

Variable scope

Like variables in programming languages, the variables in FOL have a <u>scope</u> determined by the quantifiers

Lexical scope for variables

$P(x) \wedge \exists x[P(x) \vee Q(x)]$

free bound occurrences of variables

A <u>sentence</u>: wff with no free variables (closed)

Substitution:

$\alpha[v/t]$ means α with all free occurrences of the v replaced by term t

Note: written α^{v}_{t} elsewhere (and in book)

Also: $\alpha[t_1,...,t_n]$ means $\alpha[v_1/t_1,...,v_n/t_n]$

Semantics

How to interpret sentences?

- what do sentences claim about the world?
- what does believing one amount to?

Without answers, cannot use sentences to represent knowledge

Problem:

cannot fully specify interpretation of sentences because non-logical symbols reach outside the language

So:

make clear dependence of interpretation on non-logical symbols

Logical interpretation:

specification of how to understand predicate and function symbols

Can be complex!

DemocraticCountry, IsABetterJudgeOfCharacterThan, favouriteIceCreamFlavourOf, puddleOfWater27

The simple case

There are objects.

some satisfy predicate P; some do not

Each interpretation settles <u>extension</u> of P.

borderline cases ruled in separate interpretations

Each interpretation assigns to function f a mapping from objects to objects.

functions always well-defined and single-valued

The FOL assumption:

this is all you need to know about the non-logical symbols to understand which sentences of FOL are true or false

In other words, given a specification of

» what objects there are

» which of them satisfy P

» what mapping is denoted by f

it will be possible to say which sentences of FOL are true

Interpretations

Two parts: $\mathfrak{I} = \langle D, I \rangle$

D is the domain of discourse

can be *any* non-empty set

not just formal / mathematical objects

e.g. people, tables, numbers, sentences, unicorns, chunks of peanut butter, situations, the universe

I is an interpretation mapping

If P is a predicate symbol of arity n,

$I[P] \subseteq D \times D \times \ldots \times D$

an n-ary relation over D

If f is a function symbol of arity n,

$I[f] \in [D \times D \times \ldots \times D \rightarrow D]$

an n-ary function over D

for propositional symbols,

$I[p] = \{\}$ or $I[p] = \{\langle\rangle\}$

for constants, $I[c] \in D$

In propositional case, convenient to assume

$\mathfrak{I} = I \in$ [*prop. symbols* $\rightarrow$ {true, false}]

Denotation

In terms of interpretation $\mathfrak{I}$, terms will denote elements of the domain D.

will write element as $\|t\|_{\mathfrak{I}}$

For terms with variables, the denotation depends on the values of variables

will write as $\|t\|_{\mathfrak{I},\mu}$

where $\mu \in$ [*Variables* $\rightarrow D$],
called a variable assignment

Rules of interpretation:

1. $\|v\|_{\mathfrak{I},\mu} = \mu(v)$.
2. $\| f(t_1, t_2, \ldots, t_n) \|_{\mathfrak{I},\mu} = H(d_1, d_2, \ldots, d_n)$

   where $H = I[f]$

   and $d_i = \|t_i\|_{\mathfrak{I},\mu}$, recursively

Satisfaction

In terms of an interpretation $\mathfrak{I}$, sentences of FOL will be either true or false.

Formulas with free variables will be true for some values of the free variables and false for others.

Notation:

will write as $\mathfrak{I},\mu \models \alpha$ "α is satisfied by $\mathfrak{I}$ and μ"

where $\mu \in$ [*Variables* $\rightarrow D$], as before

or $\mathfrak{I} \models \alpha$, when α is a sentence

"α is true under interpretation $\mathfrak{I}$"

or $\mathfrak{I} \models S$, when S is a set of sentences

"the elements of S are true under interpretation $\mathfrak{I}$"

And now the definition...

Rules of interpretation

1. $\mathfrak{I},\mu \models P(t_1, t_2, ..., t_n)$ iff $\langle d_1, d_2, ..., d_n \rangle \in R$
   where $R = I[P]$
   and $d_i = \| t_i \|_{\mathfrak{I},\mu}$, as on denotation slide
2. $\mathfrak{I},\mu \models (t_1 = t_2)$ iff $\| t_1 \|_{\mathfrak{I},\mu}$ is the same as $\| t_2 \|_{\mathfrak{I},\mu}$
3. $\mathfrak{I},\mu \models \neg\alpha$ iff $\mathfrak{I},\mu \not\models \alpha$
4. $\mathfrak{I},\mu \models (\alpha\wedge\beta)$ iff $\mathfrak{I},\mu \models \alpha$ and $\mathfrak{I},\mu \models \beta$
5. $\mathfrak{I},\mu \models (\alpha\vee\beta)$ iff $\mathfrak{I},\mu \models \alpha$ or $\mathfrak{I},\mu \models \beta$
6. $\mathfrak{I},\mu \models \exists v\alpha$ iff for some $d \in D$, $\mathfrak{I},\mu\{d;v\} \models \alpha$
7. $\mathfrak{I},\mu \models \forall v\alpha$ iff for all $d \in D$, $\mathfrak{I},\mu\{d;v\} \models \alpha$
   where $\mu\{d;v\}$ is just like μ, except that $\mu(v)=d$.

For propositional subset:

$\mathfrak{I} \models p$ iff $I[p] \neq \{\}$ and the rest as above

Entailment defined

Semantic rules of interpretation tell us how to understand all wffs in terms of specification for non-logical symbols.

But some connections among sentences are independent of the non-logical symbols involved.

e.g. If α is true under $\mathfrak{I}$, then so is $\neg(\beta \wedge \neg\alpha)$,
no matter what $\mathfrak{I}$ is, why α is true, what β is, ...

$S \models \alpha$ iff for every $\mathfrak{I}$, if $\mathfrak{I} \models S$ then $\mathfrak{I} \models \alpha$.

Say that *S* <u>entails</u> α or α is a <u>logical consequence</u> of *S*:

In other words: for no $\mathfrak{I}$, $\mathfrak{I} \models S \cup \{\neg\alpha\}$. $S \cup \{\neg\alpha\}$ is <u>unsatisfiable</u>

Special case when *S* is empty: $\models \alpha$ iff for every $\mathfrak{I}$, $\mathfrak{I} \models \alpha$.

Say that α is <u>valid</u>.

Note: $\{\alpha_1, \alpha_2, ..., \alpha_n\} \models \alpha$ iff $\models (\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_n) \supset \alpha$

finite entailment reduces to validity

Why do we care?

We do not have access to user-intended interpretation of non-logical symbols

But, with <u>entailment</u>, we know that if *S* is true in the intended interpretation, then so is α.

If the user's view has the world satisfying *S*, then it must also satisfy α.

There may be other sentences true also; but α is logically guaranteed.

So what about ordinary reasoning?

Dog(fido) ➠ Mammal(fido) ??

Not entailment!

There are logical interpretations where $I[\text{Dog}] \not\subset I[\text{Mammal}]$

Key idea of KR:

include such connections <u>explicitly</u> in *S*
$\forall x[\text{Dog}(x) \supset \text{Mammal}(x)]$
Get: $S \cup \{\text{Dog(fido)}\} \models \text{Mammal(fido)}$

the rest is just details...

Knowledge bases

KB is set of sentences

explicit statement of sentences believed (including any assumed connections among non-logical symbols)

KB $\models \alpha$ $\quad\alpha$ is a further consequence of what is believed

- explicit knowledge: KB
- implicit knowledge: $\{ \alpha \mid \text{KB} \models \alpha \}$

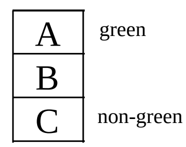
Often non trivial: explicit ➠ implicit

Example:

Three blocks stacked.

Top one is green.

Bottom one is not green.



Is there a green block directly on top of a non-green block?

A formalization

S = $\{\text{On(a,b)}, \ \text{On(b,c)}, \ \text{Green(a)}, \ \neg\text{Green(c)}\}$

all that is required

$\alpha = \exists x \exists y[\text{Green}(x) \land \neg\text{Green}(y) \land \text{On}(x,y)]$

Claim: $S \models \alpha$

Proof:

Let $\Im$ be any interpretation such that $\Im \models S$.

Case 1: $\Im \models \text{Green(b)}$.

$\therefore \ \Im \models \text{Green(b)} \land \neg\text{Green(c)} \land \text{On(b,c)}$.

$\therefore \ \Im \models \alpha$

Case 2: $\Im \not\models \text{Green(b)}$.

$\therefore \ \Im \models \neg\text{Green(b)}$

$\therefore \ \Im \models \text{Green(a)} \land \neg\text{Green(b)} \land \text{On(a,b)}$.

$\therefore \ \Im \models \alpha$

Either way, for any $\Im$, if $\Im \models S$ then $\Im \models \alpha$.

So $S \models \alpha$. QED

Knowledge-based system

Start with (large) KB representing what is explicitly known

e.g. what the system has been told or has learned

Want to influence behaviour based on what is implicit in the KB (or as close as possible)

Requires reasoning

deductive inference:

process of calculating entailments of KB

i.e given KB and any α, determine if KB $\models \alpha$

Process is sound if whenever it produces α, then KB $\models \alpha$

does not allow for plausible assumptions that may be true in the intended interpretation

Process is complete if whenever KB $\models \alpha$, it produces α

does not allow for process to miss some α or be unable to determine the status of α

